Math 121 3.2 – 2nd Graphing Using First & Second Derivatives

Objectives

1) Use second derivative test for relative extrema.
2) Know limitations of the second derivative test.
3) Know the difference between the first derivative test and the second derivative test.
4) Interpret inflection points in applications

Recall: Relative extrema (max or min) occur at critical numbers which we obtain from $f'(x)=0$ or $f'(x)$ undefined.
THIS DOES NOT CHANGE.

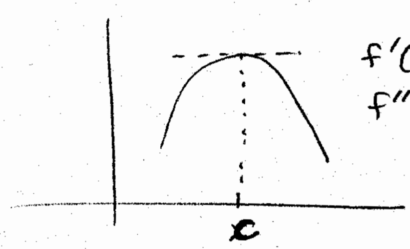
But determining if a critical number gives a relative min or max or neither required a sign chart of $f'(x)$ to see if the sign of f' changed (+) to (−) (MAX)

(−) to (+) MIN

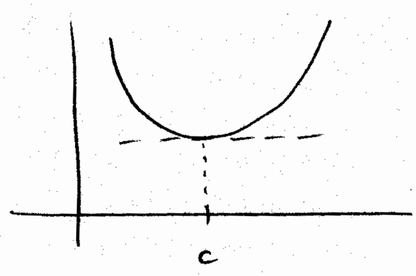
(+) to (+), (−) to (−) } no change ⇒ neither max nor min.

This sign chart for $f'(x)$ was called the first derivative test.

We can determine if a critical number gives a relative max or min another way, by using the second derivative.
Because f'' tells us concavity, we use it differently — we don't need an entire sign chart, only the concavity at a critical value.



$f'(c)=0$ gives a horizontal tangent }
$f''(c)<0$ says concave down } together this means relative maximum



$f'(c)=0$ gives a horizontal tangent }
$f''(c)>0$ means concave up } together this means relative minimum

① Use the second derivative test to determine the relative extrema of $f(x) = x^3 - 9x^2 + 24x$

Step 1: Find $f'(x)$

$$f'(x) = 3x^2 - 18x + 24$$

Step 2: Find critical values where $f'(x) = 0$.

$$3x^2 - 18x + 24 = 0$$
$$3(x^2 - 6x + 8) = 0$$
$$3(x-4)(x-2) = 0$$
$$x = 4 \qquad x = 2$$

Step 3: Find $f''(x)$

$$f''(x) = 6x - 18$$

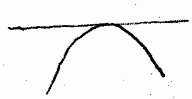
Step 4: Find $f''(c)$ for each critical value

$f''(4) = 6(4) - 18 = 6 > 0$

concave up
horizontal tangent

relative minimum at $x = 4$.
$f(4) = 4^3 - 9(4)^2 + 24(4) = 16$ y coord.

$f''(2) = 6(2) - 18 = -6 < 0$



concave down
horizontal tangent

relative maximum @ $x = 2$
$f(2) = 2^3 - 9(2)^2 + 24(2) = 20$ y coord

rel min $(4, 16)$ rel max $(2, 20)$

The advantages of the second derivative test is
a) evaluating in f'' is usually easier arithmetic
b) we evaluate only once to find our conclusion

The disadvantages of the second derivative test...
a) if $f''(c) = 0$ we cannot determine any conclusion
This is called the indeterminate case.

Attempt to find the relative extrema using the second derivative test. Then graph on GC to identify max or min.

(2) $f(x)=x^3$

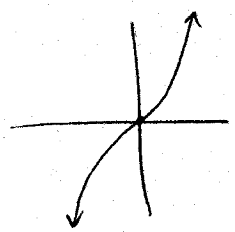
$f'(x)=3x^2$

$f'(x)=0 \quad 3x^2=0 \quad x=0$ critical number

$f''(x)=6x$

$f''(0)=6(0)=0$! cannot determine max or min if $f''(c)=0$!!

GC



$x=0$ is neither a max nor a min.

(3) $f(x)=x^4$

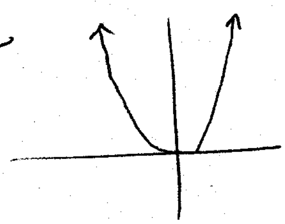
$f'(x)=4x^3$

$f'(x)=0 \quad 4x^3=0 \quad x=0$ critical number

$f''(x)=12x^2$

$f''(0)=0$! cannot determine max or min if $f''(c)=0$!!

GC



$x=0$ is a relative min.

(4) $f(x)=-x^4$

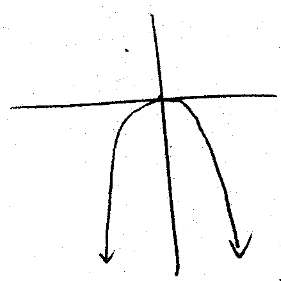
$f'(x)=-4x^3$

$f'(x)=0 \quad -4x^3=0 \quad x=0$ critical number

$f''(x)=-12x^2$

$f''(0)=0$!

GC



$x=0$ is a relative max.

These three examples all have $f''(c)=0$, but in each case we get a different outcome.
So $f''(c)$ does not tell us anything.

Second Derivative Test

If $x=c$ is a critical number $f'(c)=0$

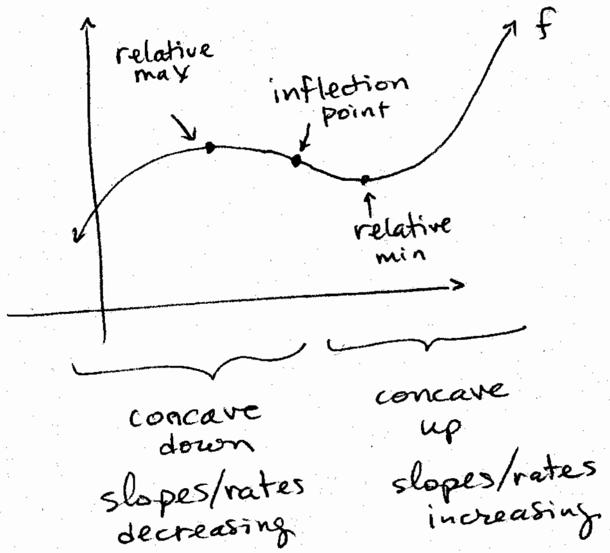
and $f''(c)>0$ — $x=c$ is the location of a relative maximum

OR $f''(c)<0$ — $x=c$ is the location of a relative minimum

If $f''(c)=0$ the test is indeterminate

If $f'(c)$ undefined, then $f''(c)$ is also undefined and the test is indeterminate.

Interpreting an inflection point:



If f measures a desirable quantity

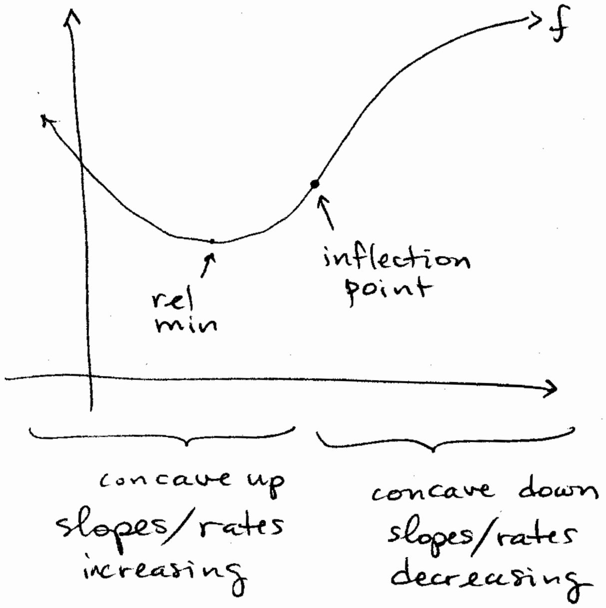
ex:
- profit
- revenue
- drug response
- social status
- sales

and the inflection point is a change from c-down to c-up, then the situation is improving, and the inflection point is the first sign of this improvement.

If f measures an undesirable quantity

ex:
- # cases of a disease
- costs
- pollen count
- airplane accidents

and the inflection point is a change from c-down to c-up, then the situation is worsening, and the inflection point is the first sign of bad things to come.



If f measures a desirable quantity and the inflection point is a change from c-up to c-down, then the situation is worsening, and the inflection point is the first sign of these bad things to come.

If f measures an undesirable quantity and the inflection point is a change from c-up to c-down, then the situation is improving, and the inflection point is the first sign of improvement.

Summary

	Inflection Point concave down to concave up	Inflection Point concave up to concave down
f desirable	first sign of improvement	first sign of worsening situation
f undesirable	first sign of worsening situation	first sign of improvement

(5) A company's annual profit after x years is $f(x) = x^3 - 9x^2 + 24x$ in millions dollars, for $x \geq 0$. a) Find and interpret the inflection point(s).

$f'(x) = 3x^2 - 18x + 24$

$f''(x) = 6x - 18$

$f''(x) = 0$

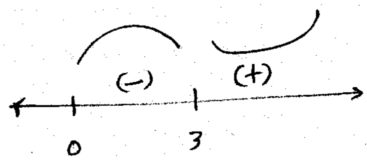
$6x - 18 = 0$

$6x = 18$

$x = 3$.

$f(3) = 3^3 - 9(3)^2 + 24(3) = 18$

sign chart f''



changes from concave down (rate of change decreasing to rate of change increasing).

$\boxed{\text{After 3 years, profits show the first sign of improving their rate of change.}}$

b) Find and interpret the relative extrema.

$f'(x) = 0$

$3x^2 - 18x + 24 = 0$

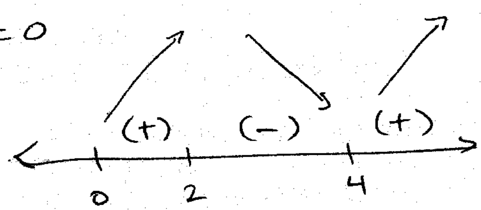
$3(x^2 - 6x + 8) = 0$

$3(x-4)(x-2) = 0$

$x = 4, \ x = 2$

$f(2) = 2^3 - 9(2)^2 + 24(2)$
$= 20$

sign chart f'



$f(4) = 4^3 - 9(4)^2 + 24(4)$
$= 16$

$\boxed{\text{Profits increased up to a maximum \$20,000,000 after 2 years, but decreased until 4 years, when they were \$16,000,000.}}$

Though profits are increasing from $x=0$ to $x=2$, the rate with which they increase was decreasing.

This decrease continued until after $t=2$, when the rates themselves showed decrease in profits.

This first sign that the downward trend in profits was ending came at $x=3$, the inflection point.

Notice that profits f are always positive → the company is always making a profit!

c) Sketch the graph of f.

f': (+) (−) (−) (+)

f'': (−) (−) (+) (+)

0 2 3 4

increasing	dec	dec	inc
c-down	cdown	cup	cup